

Concurrent Quantum Separation Logic for Fine-Grained Parallelism

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Abstract

A promising approach to efficient quantum computation is to execute subroutines in *parallel* at a fine-grained level. While such parallelism is subject to tricky bugs, there was no quantum program logic that could modularly verify the correctness of such parallelism.

To overcome this situation, we propose novel concurrent quantum separation logic that can modularly reason about quantum programs under fine-grained parallelism. Our logic enables flexible reasoning about *quantum superposition* via new proof rules for *linearly combining* Hoare triples. Also, our logic introduces fractional tokens for *sharing* the same qubits between parallel subroutines, introducing new reasoning rules for *promoting* partial ownership into full ownership by *atomicity*. We demonstrate the effectiveness of our logic by verifying a non-trivial parallelized quantum program.

1 Introduction

Today, quantum computers are steadily becoming larger, and interest in realizing efficient quantum computation has grown. A promising approach to that is to execute subroutines in *parallel* at a fine-grained level [Gidney and Ekerå 2021; Häner et al. 2022]. One practical goal would be the automated parallelization of quantum programs, statically by the compiler or dynamically by the runtime. While such parallelism is subject to tricky bugs, there was no scalable quantum program logic that could verify the correctness of such parallelism. To tackle state mutation under concurrency, *concurrent separation logic* [O’Hearn 2004; Brookes 2004; Brookes and O’Hearn 2016] should be a great fit, but existing quantum separation logic [Zhou et al. 2021; Le et al. 2022; Su et al. 2024] unfortunately supported neither concurrency nor sharing of qubits, which is crucial for fine-grained parallelism.

To overcome this situation, we propose novel concurrent quantum separation logic that can modularly

reason about quantum programs under fine-grained parallelism. Our logic is particularly new in the following two points. First, unlike the classical setting and existing quantum separation logic, our logic enables flexible reasoning about *quantum superposition* via new proof rules for *linearly combining* Hoare triples. Second, our logic features a *fractional* quantum points-to token $\bar{q} \mapsto^r |\psi\rangle$, which can be *shared* between parallel subroutines, introducing new reasoning rules for *promoting* partial ownership into full ownership by *atomicity*. We demonstrate the effectiveness of our logic by verifying a non-trivial parallelized quantum program.

2 Target Quantum Language

Our target quantum language has the following syntax:

$$\begin{aligned} \text{Exp} \ni e &::= q \mid \ell \mid n \mid () \mid \text{op}(\bar{e}) \mid x \\ &\mid \text{let } x = e_1 \text{ in } e_2 \mid \text{if } e_1 \{ e_2 \} \text{ else } \{ e_3 \} \\ &\mid \text{while } e_1 \{ e_2 \} \mid e_1 \parallel e_2 \mid \text{atomic } \{ e \} \\ &\mid \text{qalloc} \mid \text{qfree } e \mid U[\bar{e}] \mid \text{mkref } e \mid !e \mid e_1 \leftarrow e_2 \\ \text{Val} \ni v &::= q \mid \ell \mid n \mid () \mid (v, v') \quad e_1; e_2 \triangleq \text{let } _ = e_1 \text{ in } e_2 \end{aligned}$$

We write $q \in \text{Qname}$ for a qubit (name), $\ell \in \text{Loc}$ for a heap location, $n \in \mathbb{Z}$ for an integer, op for pure operators such as $+$. For concurrency, aside from the standard *parallel execution* $e_1 \parallel e_2$, we introduce *atomic execution* $\text{atomic } \{ e \}$, which excludes interruption from other processes while executing e . Qubit and heap operations are pretty standard. Quantum measurements are not yet part of our language, and handling their stochastic behavior remains future work (see §6). Please refer to [Appendix A](#) for the operational semantics.

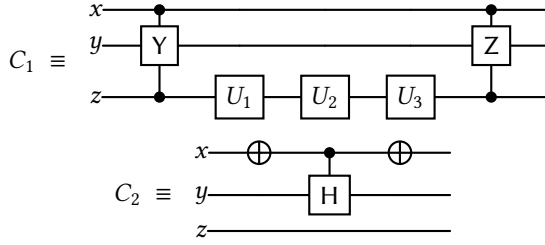
3 Motivating Example

As a running example, we consider the following parallelized quantum program:

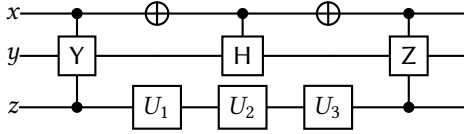
$$\begin{aligned} &\text{CCY}[x, z, y]; U_1[z]; U_2[z]; U_3[z]; \text{CCZ}[x, z, y] \\ &\parallel \text{atomic } \{ X[x]; \text{CH}[x, y]; X[x] \} \end{aligned} \quad (*)$$

* This is an extended abstract of the talk presented at Theory and Practice of Static Analysis (TPSA) 2025.

It executes in parallel two processes, which can be illustrated as the following circuits:



The two processes may be executed sequentially (i.e., $C_1; C_2$ or $C_2; C_1$), or process 2 may be executed during the execution of process 1 as follows:



It is easy to see that the results are the same across these execution patterns.

Our goal is to prove the correctness of such a parallelized quantum program: regardless of the execution order, the program should consistently reach the same final state. This enables a compiler to safely reorder the execution sequence of concurrent circuits, provided it adheres to the program's semantics. However, modularly proving the correctness of a parallelized quantum program like (*) presents the following two challenges:

Challenge 1 Multiple processes may write to the same qubit. In the program (*), both processes write to the shared qubit y in parallel. This is naively a race condition but actually safe thanks to the commutativity of the gates.

Challenge 2 Atomic execution should be treated specially. In (*), Process 2 writes to x but then reverts its value within an atomic execution, and thus does not interfere with Process 1.

4 Our Quantum Separation Logic

Now we present our core contribution, the concurrent quantum separation logic for fine-grained parallelism. Please refer to [Appendix B](#) for a complete list of proof rules and [Appendix C](#) for the semantic model.

Propositions. Propositions are as follows:

$$\begin{aligned}
 SLProp \ni P, Q, R ::= & p \mid P \wedge Q \mid P \vee Q \mid P \rightarrow Q \\
 & \mid \forall x. P_x \mid \exists x. P_x \mid \text{emp} \mid P * Q \mid P \multimap Q \\
 & \mid \bar{q} \xrightarrow{r} |\psi\rangle \mid [q]_r \mid \ell \xrightarrow{r} v \\
 & \bar{q} \mapsto |\psi\rangle \triangleq \bar{q} \xrightarrow{1} |\psi\rangle \quad \ell \mapsto v \triangleq \ell \xrightarrow{1} v
 \end{aligned}$$

We have the standard connectives from separation logic, including the separating conjunction $P * Q$. Any pure proposition $p \in Prop$ can be embedded.

Notably, we introduce the *fractional quantum points-to token* $\bar{q} \xrightarrow{r} |\psi\rangle$, a novel proposition that asserts with a fraction $r \in (0, 1]$ that the current pure state of the qubits $\bar{q} = q_1, \dots, q_n$ is $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$, analogously to the classical points-to token $\ell \xrightarrow{r} v$ [Bornat et al. 2005]. The model for this machinery is non-trivial and trickier than the classical setting; please see [Appendix C](#).

We also introduce the *qubit token* $[q]_r$, which asserts with a fraction r that the qubit q is allocated and not freed. It is useful for sharing *dirty qubits*, qubits without state information, between parallel subroutines. We use the shorthand $[\bar{q}]_r \triangleq *_i [q_i]_r$ for $\bar{q} = q_1, \dots, q_n$.

We have the following proof rules for quantum ownership, as naturally expected:

$$\begin{aligned}
 (\bar{q}, \bar{q}') \mapsto |\psi\rangle |\psi'\rangle & \vdash \bar{q} \mapsto |\psi\rangle * \bar{q}' \mapsto |\psi'\rangle \\
 \bar{q} \xrightarrow{r+r'} |\psi\rangle & \vdash \bar{q} \xrightarrow{r} |\psi\rangle * \bar{q} \xrightarrow{r'} |\psi\rangle \\
 \{\text{emp}\} \text{qalloc } \{q. q \mapsto |0\rangle * [q]_1\} \\
 \{q \mapsto |0\rangle * [q]_1\} \text{qfree } q & \{\text{emp}\} \\
 \{(\bar{q}, \bar{q}') \mapsto |\psi\rangle\} U[\bar{q}] & \{(\bar{q}, \bar{q}') \mapsto (U \otimes I) |\psi\rangle\}
 \end{aligned}$$

Quantum superposition. One peculiar phenomenon in quantum computation is *quantum superposition*. While we can easily reason about classical conditional branching by case analysis between 1 and 0 on a classical bit, reasoning about controlled gates such as CX is much trickier because a qubit can be an arbitrary quantum superposition $\alpha|1\rangle + \beta|0\rangle$. To address this, we newly introduce the following rule for *linearly combining* Hoare triples, unlike existing quantum separation logic:

$$\frac{\{ \bar{q} \mapsto |\psi\rangle * P \} e \{ \bar{q} \mapsto |\varphi\rangle * Q \} \quad \{ \bar{q} \mapsto |\psi'\rangle * P \} e \{ \bar{q} \mapsto |\varphi'\rangle * Q \} \quad P, Q: \text{precise}}{\{ \bar{q} \mapsto (\alpha|\psi\rangle + \beta|\psi'\rangle) * P \} e \{ \bar{q} \mapsto (\alpha|\varphi\rangle + \beta|\varphi'\rangle) * Q \}}$$

QPTTO-LINCOMB

The exactness judgment $P: \text{precise}$ means that the SL proposition P is satisfied by a unique (or no) resource (for example, $[x]_1: \text{precise}$ and $\perp: \text{precise}$ hold but $(\exists q. [x]_q): \text{precise}$ does not hold). At a high level, this performs a ‘case analysis’ over the bases $|\psi\rangle$ and $|\psi'\rangle$ of the input pure state $\alpha|\psi\rangle + \beta|\psi'\rangle$. This actually solves Challenge 1 in §3, because the ‘case analysis’ over x eliminates the spurious race condition. See the discussion in §5 for details.

Concurrency. We have the following for concurrency:

$$\frac{\frac{\{P\} e \{v. Q_v\} \quad \{P'\} e' \{v'. Q'_{v'}\}}{\{P * P'\} e \parallel e' \{(v, v'). Q_v * Q'_{v'}\}} \text{PARALLEL}}{\frac{\{P\} e \{v. Q_v\}}{\{P\} \text{atomic} \{e\} \{v. Q_v\}} \text{ATOMIC}}$$

Parallel execution $e \parallel e'$ admits the standard proof rule with the separating conjunction. For atomic execution $\text{atomic} \{e\}$, we simply verify the Hoare triple over e , with *promotion by atomicity* explained below.

Promotion by atomicity. We say an expression e is atomic if e can take only one step. In particular, $\text{atomic} \{e\}$ and $U[\bar{q}]$ are atomic. For atomic expressions, we can use the following rules for *promotion*:

$$\frac{\frac{e \text{ is atomic} \quad P : \text{out } \bar{q}}{\{\bar{q} \mapsto |\psi\rangle * [\bar{q}]_1 * P\} e \{v. \bar{q} \mapsto |\psi\rangle * [\bar{q}]_1 * Q_v\}}}{\{\bar{q} \xrightarrow{r} |\psi\rangle * P\} e \{v. \bar{q} \xrightarrow{r} |\psi\rangle * Q_v\}} \text{QPTTO-PROMOTE}$$

$$\frac{\forall |\psi\rangle. \{q \mapsto |\psi\rangle * [q]_1 * P\} e \{v. q \mapsto |\psi\rangle * [q]_1 * Q_v\}}{\{[q]_r * P\} e \{v. [q]_r * Q_v\}} \text{QTOK-PROMOTE}$$

By exploiting atomicity, partial ownership of a fractional points-to $\bar{q} \xrightarrow{r} |\psi\rangle$ (**QPTTO-PROMOTE**) or qubit $[q]_r$ (**QTOK-PROMOTE**) token can be promoted into the full points-to and qubit tokens. Here, the judgment $P : \text{out } \bar{q}$ means that the proposition P does not own any ownership over the qubits $\bar{q}$. For example, $q' \mapsto |\psi\rangle : \text{out } \bar{q}$ holds if $q' \notin \{\bar{q}\}$. Promotion solves Challenge 2 in §3, enabling temporary writes in an atomic execution.

5 Verification of the Motivating Example

Now we demonstrate how our separation logic can verify the concurrent quantum (*) presented in §3.

First, we apply **QPTTO-LINCOMB** to decompose the superposition of x 's pure state $|\chi\rangle = \alpha|1\rangle + \beta|0\rangle$:

$$\begin{aligned} & \{(x, y, z) \mapsto |\chi\rangle |\psi\rangle |\varphi\rangle * [y]_1\} \\ & \{(x, y, z) \mapsto |1\rangle |\psi\rangle |\varphi\rangle * [y]_1\} \{(x, y, z) \mapsto |0\rangle |\psi\rangle |\varphi\rangle * [y]_1\} \\ & \{(x \xrightarrow{1/2} |1\rangle * (y, z) \mapsto |\psi\rangle |\varphi\rangle) * (x \xrightarrow{1/2} |1\rangle * [y]_1)\} \\ & \{(x \xrightarrow{1/2} |0\rangle * [y]_1 * z \mapsto |\varphi\rangle) * (x \xrightarrow{1/2} |0\rangle * y \mapsto |\psi\rangle)\} \\ & \text{CCY}[x, z, y]; U_1[z]; U_2[z]; U_3[z]; \text{CCZ}[x, z, y] \\ & \parallel \text{atomic} \{X[x]; \text{CH}[x, y]; X[x]\} \\ & \{(x \xrightarrow{1/2} |1\rangle * (y, z) \mapsto U'_+(\psi) |\varphi\rangle) * (x \xrightarrow{1/2} |1\rangle * [y]_1)\} \\ & \{(x \xrightarrow{1/2} |0\rangle * [y]_1 * z \mapsto U_+ |\varphi\rangle) * (x \xrightarrow{1/2} |0\rangle * y \mapsto H |\psi\rangle)\} \end{aligned}$$

$$\begin{aligned} & \{x \mapsto |1\rangle * (y, z) \mapsto U'_+(\psi) |\varphi\rangle * [y]_1\} \\ & \{x \mapsto |0\rangle * y \mapsto H |\psi\rangle * z \mapsto U_+ |\varphi\rangle * [y]_1\} \\ & \{(x, y, z) \mapsto \text{If}(U'_+, H \otimes U_+) (|\chi\rangle |\psi\rangle |\varphi\rangle) * [y]_1\} \end{aligned}$$

We use the shorthand $U_+ \triangleq U_3 U_2 U_1$, $U'_+ \triangleq \text{CZ}_{zy} (I \otimes U_+) \text{CY}_{zy}$ and define the linear map $\text{If}(U_1, U_0)$ by $\text{If}(U_1, U_0) (|b\rangle |\omega\rangle) \triangleq |b\rangle \cdot U_b |\omega\rangle$ for $b \in \{1, 0\}$. The final postcondition says that, regardless of execution scheduling, the final state of the qubits x, y, z is always set to $\text{If}(U'_+, H \otimes U_+) (|\chi\rangle |\psi\rangle |\varphi\rangle)$. By **QPTTO-LINCOMB**, the verification boils down to the two cases where the qubit x stores $|1\rangle$ (marked green) and $|0\rangle$ (marked blue). Here, $\{P_1\} \{P_2\} e \{Q_1\} \{Q_2\}$ indicates that both $\{P_1\} e \{Q_1\}$ and $\{P_2\} e \{Q_2\}$ hold. Notably, how to split the ownership between two processes in **PARALLEL** can depend on whether x stores $|1\rangle$ or $|0\rangle$. More concretely, here the full points-to token $y \mapsto |\psi\rangle$ over the qubit y is given to Process 1 if x stores $|1\rangle$ and to Process 2 if x stores $|0\rangle$. This solves Challenge 1.

We can verify Process 1 as follows:

$$\begin{aligned} & \{x \xrightarrow{1/2} |1\rangle * (y, z) \mapsto |\psi\rangle |\varphi\rangle\} \{x \xrightarrow{1/2} |0\rangle * [y]_1 * z \mapsto |\varphi\rangle\} \\ & \text{CCY}[x, z, y] \\ & \{x \xrightarrow{1/2} |1\rangle * (y, z) \mapsto \text{CY}_{zy} (|\psi\rangle |\varphi\rangle)\} \{x \xrightarrow{1/2} |0\rangle * [y]_1 * z \mapsto |\varphi\rangle\} \\ & U_1[z]; U_2[z]; U_3[z]; \\ & \{x \xrightarrow{1/2} |1\rangle * (y, z) \mapsto U'_+(\psi) |\varphi\rangle\} \{x \xrightarrow{1/2} |0\rangle * [y]_1 * z \mapsto U_+ |\varphi\rangle\} \\ & \text{CCZ}[x, z, y] \\ & \{x \xrightarrow{1/2} |1\rangle * (y, z) \mapsto U'_+(\psi) |\varphi\rangle\} \{x \xrightarrow{1/2} |0\rangle * [y]_1 * z \mapsto U_+ |\varphi\rangle\} \end{aligned}$$

We use the shorthand $U'_+ \triangleq (I \otimes U_+) \text{CY}_{zy}$. Notably, the qubit token $[y]_1$ suffices for performing $\text{CCX}[x, y, z]$ and $\text{CCY}[x, z, y]$ in the case where x stores $|0\rangle$.

We can verify Process 2 as follows:

$$\begin{aligned} & \{x \xrightarrow{1/2} |1\rangle * [y]_1\} \{x \xrightarrow{1/2} |0\rangle * y \mapsto |\psi\rangle\} \\ & \text{atomic} \{ \{x \mapsto |1\rangle * [y]_1\} \{x \mapsto |0\rangle * y \mapsto |\psi\rangle\} X[x]; \\ & \quad \{x \mapsto |0\rangle * [y]_1\} \{x \mapsto |1\rangle * y \mapsto |\psi\rangle\} \text{CH}[x, y]; \\ & \quad \{x \mapsto |0\rangle * [y]_1\} \{x \mapsto |1\rangle * y \mapsto H |\psi\rangle\} X[x]; \\ & \quad \{x \mapsto |1\rangle * [y]_1\} \{x \mapsto |0\rangle * y \mapsto H |\psi\rangle\} \} \\ & \{x \xrightarrow{1/2} |1\rangle * [y]_1\} \{x \xrightarrow{1/2} |0\rangle * y \mapsto H |\psi\rangle\} \end{aligned}$$

Remarkably, we can *promote* the partial points-to token $x \xrightarrow{1/2} |\chi\rangle$ into the full points-to token $x \mapsto |\chi\rangle$ by exploiting the *atomicity* of atomic , because the value of x is finally restored. This solves Challenge 2.

6 Future Work

We plan to apply our separation logic to verify more practical quantum programs. We are particularly interested in verifying optimization techniques that involve

advanced concurrency, entangled copying, or uncomputation [Bichsel et al. 2020]. Furthermore, we aim to develop new methods for automatically parallelizing quantum programs via ownership analysis based on our separation logic. Also, future work remains in extending our concurrent quantum separation logic to support measurement. A fundamental challenge is how to extend proof rules like **QPTTO-LINCOMB** for superposition to that setting. For precise analysis of probabilistic distribution, an approach like outcome separation logic [Zilberstein et al. 2024] may be a good fit.

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A Operational Semantics of the Target Quantum Language

Here we present the operational semantics of the target quantum language of § 2.

Transition label. We use the following transition labels in our operational semantics:

$$Label \ni L ::= \text{qalloc} = q \mid \text{qfree } q \mid U[\bar{q}] \mid \text{mkref } v = \ell \mid !\ell = v \mid \ell \leftarrow v$$

Labeled transition over expressions. Evaluation contexts have the following form:

$$K ::= \cdot \mid \text{op}(\bar{v}, K, \bar{e}) \mid \text{let } x = K \text{ in } e \mid \text{if } K \{e_1\} \text{ else } \{e_2\} \mid K \parallel e \mid e \parallel K \\ \mid \text{qfree } K \mid U[\bar{v}, K, \bar{e}] \mid \text{mkref } K \mid !K \mid K \leftarrow e \mid v \leftarrow K$$

The labeled transition $e \xrightarrow{\bar{L}} e'$ (where $\bar{L} \in Label^*$ is a finite sequence of transition labels) is inductively defined by the following rules:

$$\frac{e \xrightarrow{\bar{L}} e'}{K[e] \xrightarrow{\bar{L}} K[e']} \quad \frac{\text{op}(\bar{v}) = v'}{\text{op}(\bar{v}) \rightarrow v'} \quad \text{let } x = v \text{ in } e \rightarrow e[v/x] \\ \text{if } b \{e_1\} \text{ else } \{e_0\} \rightarrow e_b \quad \text{while } e_1 \{e_2\} \rightarrow \text{if } e_1 \{e_2; \text{while } e_1 \{e_2\}\} \\ \frac{e_1 \xrightarrow{\bar{L}_1} e_2 \xrightarrow{\bar{L}_2} \dots \xrightarrow{\bar{L}_n} v}{\text{atomic } \{e_1\} \xrightarrow{\bar{L}_1, \bar{L}_2, \dots, \bar{L}_n} v} \\ \text{qalloc} \xrightarrow{\text{qalloc} = q} q \quad \text{qfree } q \xrightarrow{\text{qfree } q} () \quad U[\bar{q}] \xrightarrow{U[\bar{q}]} () \\ \text{mkref } v \xrightarrow{\text{mkref } v = \ell} \ell \quad !\ell \xrightarrow{!\ell = v} v \quad \ell \leftarrow v \xrightarrow{\ell \leftarrow v} ()$$

Here we use the shorthand $\text{if } e_1 \{e_2\} \triangleq \text{if } e_1 \{e_2\} \text{ else } \{()\}$.

The labeled infinite transition $e \xrightarrow{\bar{L}} \infty$ (where $\bar{L} \in Label^* \cup Label^\omega$ is a *possibly infinite* sequence of transition labels) is defined by the following rules:

$$\frac{e_1 \xrightarrow{\bar{L}_1} e_2 \xrightarrow{\bar{L}_2} \dots \xrightarrow{\bar{L}_{k-1}} e_k \xrightarrow{\bar{L}_k} \dots}{K[\text{atomic } \{e_1\}] \xrightarrow{\bar{L}_1, \bar{L}_2, \dots, \bar{L}_{k-1}, \bar{L}_k, \dots} \infty} \quad \frac{e_1 \xrightarrow{\bar{L}_1} e_2 \xrightarrow{\bar{L}_2} \dots \xrightarrow{\bar{L}_n} e_{n+1} \xrightarrow{\bar{L}'} \infty}{K[\text{atomic } \{e_1\}] \xrightarrow{\bar{L}_1, \bar{L}_2, \dots, \bar{L}_n, \bar{L}'} \infty}$$

Atomicity. We say an expression e is atomic if $e \xrightarrow{\bar{L}} e'$ entails $e' \in Val$ for any expression e' and labels $\bar{L}$. The following rules hold:

$$\text{atomic } \{e\} \text{ is atomic} \quad \text{qalloc is atomic} \quad \text{qfree } v \text{ is atomic} \quad U[\bar{v}] \text{ is atomic} \\ \text{mkref } v \text{ is atomic} \quad !v \text{ is atomic} \quad v \leftarrow v' \text{ is atomic}$$

Labeled transition over global states. The global state we consider is as follows:

$$\text{Quantum memory } M = (qs, |\psi\rangle) \in Qmem \triangleq \coprod_{qs \in \text{Pow}_{\text{fin}} Qname} (\mathbb{C}^2)^{\otimes qs}$$

$$\text{Heap memory } H \in Heap \triangleq Loc \xrightarrow{\text{fin}} Val$$

$$\text{Global state } G = (M, H) \in Glob \triangleq Qmem \times Heap$$

The labeled global state transition $G \xrightarrow{L} G'$ is defined by the following rules:

$$\frac{q \notin qs}{((qs, |\psi\rangle), H) \xrightarrow{\text{qalloc} = q} ((qs \cup \{q\}, |\psi\rangle_{qs} \otimes |0\rangle_q), H)}$$

$$\begin{array}{c}
\frac{q \notin qs}{((qs \cup \{q\}, |\psi\rangle_{qs} \otimes |0\rangle_q), H) \xrightarrow{\text{qfree } q} ((qs, |\psi\rangle), H)} \\
\frac{\bar{q} \text{ are pairwise distinct} \quad \{\bar{q}\} \cap qs \neq \emptyset}{((\{\bar{q}\} \cup qs, |\psi\rangle), H) \xrightarrow{U[\bar{q}]} ((\{\bar{q}\} \cup qs, U_{\bar{q}} |\psi\rangle), H)} \\
\frac{\ell \notin \text{dom } H}{(M, H) \xrightarrow{\text{mkref } v = \ell} (M, H\{\ell := v\})} \quad \frac{\ell \in \text{dom } H \quad v = H[\ell]}{(M, H) \xrightarrow{\ell = v} (M, H)} \quad \frac{\ell \in \text{dom } H}{(M, H) \xrightarrow{\ell \leftarrow v} (M, H\{\ell := v\})}
\end{array}$$

Expression-state transition. The expression-state transition $(e, G) \rightarrow (e', G')$ is defined by the following rule:

$$\frac{e \xrightarrow{L_1, L_2, \dots, L_n} e' \quad G_1 \xrightarrow{L_1} G_2 \xrightarrow{L_2} \dots \xrightarrow{L_n} G_{n+1}}{(e, G_1) \rightarrow (e', G_{n+1})}$$

The expression-state infinite transition $(e, G) \rightarrow \infty$ is defined by the following rule:

$$\frac{e \xrightarrow{L_1, L_2, \dots, L_{k-1}, L_k, \dots} \infty \quad G_1 \xrightarrow{L_1} G_2 \xrightarrow{L_2} \dots \xrightarrow{L_{k-1}} G_k \xrightarrow{L_k} \dots}{(e, G_1) \rightarrow \infty}$$

The reducibility $\text{red}(e, G)$ is defined by the following rule:

$$\frac{(e, G) \rightarrow (e', G')}{\text{red}(e, G)} \quad \frac{(e, G) \rightarrow \infty}{\text{red}(e, G)}$$

B Proof Rules of Our Separation Logic

Judgments. We use the following judgments:

$$P \vdash Q \quad \{P\} e \{v. Q_v\} \quad P : \text{out } a$$

As usual, we introduce the entailment judgment $P \vdash Q$ and the (partial) Hoare triple $\{P\} e \{v. Q_v\}$ (where v is the return value of the execution). We also introduce the ownership exclusion judgment $P : \text{out } a$, meaning that a proposition $P \in \text{SLProp}$ does not own a qubit or location $a \in \text{Qname} \cup \text{Loc}$.

We use the following syntax sugar:

$$\begin{aligned}
P \dashv\vdash Q &\triangleq (P \vdash Q) \wedge (Q \vdash P) & \{P\} e \{Q\} &\triangleq \{P\} e \{_ . Q\} \\
P : \text{out } \bar{a} &\triangleq \forall i. (P : \text{out } a_i)
\end{aligned}$$

General proof rules.

$$\begin{array}{c}
P \vdash P \quad \frac{P \vdash Q \quad Q \vdash R}{P \vdash R} \quad \frac{p \text{ holds}}{P \vdash p} \quad \frac{P \vdash p \quad p \text{ implies } P \vdash Q}{P \vdash Q} \\
P_1 \wedge P_2 \vdash P_i \quad \frac{R \vdash P \quad R \vdash Q}{R \vdash P \wedge Q} \quad P_i \vdash P_1 \vee P_2 \quad \frac{P \vdash R \quad Q \vdash R}{P \vee Q \vdash R} \\
(P \rightarrow Q) \wedge P \vdash Q \quad \frac{P \wedge Q \vdash R}{P \vdash Q \rightarrow R} \\
(\forall x. P_x) \vdash P_a \quad \frac{\forall x. (Q \vdash P_x)}{Q \vdash \forall x. P_x} \quad P_a \vdash (\exists x. P_x) \quad \frac{\forall x. (P_x \vdash Q)}{(\exists x. P_x) \vdash Q} \\
\frac{P \vdash Q}{P * R \vdash Q * R} \quad P * \text{emp} \dashv\vdash P \quad P * Q \dashv\vdash Q * P \quad (P * Q) * R \dashv\vdash P * (Q * R) \\
(P \multimap Q) * P \vdash Q \quad \frac{P * Q \vdash R}{P \vdash Q \multimap R}
\end{array}$$

Basic Hoare-triple proof rules.

$$\begin{array}{c}
\frac{P' \vdash P \quad \{P\} e \{v. Q_v\} \quad \forall v. (Q_v \vdash Q'_v)}{\{P'\} e \{v. Q'_v\}} \quad \frac{\{P\} e \{v. Q_v\}}{\{P * R\} e \{v. Q_v * R\}} \\
\frac{\{\perp\} e \{v. Q_v\}}{\{\exists x. P_x\} e \{v. Q_v\}} \quad \frac{\forall x. \{P_x\} e \{v. Q_v\}}{\{\exists x. P_x\} e \{v. Q_v\}} \\
\frac{\{P\} e \{v. Q_v\} \quad \forall v. \{Q_v\} K[v] \{v'. R_{v'}\}}{\{P\} K[e] \{v'. R_{v'}\}} \quad \frac{\{P\} e[v/x] \{v'. Q_{v'}\}}{\{P\} \text{let } x = v \text{ in } e \{v'. Q_{v'}\}} \\
\frac{\{P\} e_b \{v. Q_v\}}{\{P\} \text{if } b \{e_1\} \text{else } \{e_0\} \{v. Q_v\}} \quad \frac{\{P\} e_1 \{b. Q_b\} \quad \{Q_1\} e_2 \{P\}}{\{P\} \text{while } e_1 \{e_2\} \{v. Q_0\}}
\end{array}$$

Basic proof rules for concurrency.

$$\frac{\{P\} e \{v. Q_v\} \quad \{P'\} e' \{v'. Q'_{v'}\}}{\{P * P'\} e \parallel e' \{(v, v'). Q_v * Q'_{v'}\}} \quad \text{PARALLEL} \quad \frac{\{P\} e \{v. Q_v\}}{\{P\} \text{atomic } \{e\} \{v. Q_v\}} \quad \text{ATOMIC}$$

Rules for exactness.

$$\begin{array}{c}
\text{emp : precise} \quad \frac{P, Q : \text{precise}}{P * Q : \text{precise}} \quad \frac{P : \text{precise} \quad Q \vdash P}{Q : \text{precise}} \\
\bar{q} \xrightarrow{r} |\psi\rangle : \text{precise} \quad [q]_r : \text{precise} \quad \ell \xrightarrow{r} v : \text{precise}
\end{array}$$

Rules for quantum and heap ownership exclusion.

$$\begin{array}{c}
\text{emp : out } a \quad \frac{P, Q : \text{out } a}{P \vee Q, P * Q : \text{out } a} \quad \frac{\forall x. (P_x : \text{out } a)}{(\exists x. P_x) : \text{out } a} \quad \frac{P : \text{out } a \quad Q \vdash P}{Q : \text{out } a} \\
\bar{q} \xrightarrow{r} |\psi\rangle, [q]_r : \text{out } \ell \quad \frac{q \notin \{\bar{q}'\}}{\bar{q}' \xrightarrow{r} |\psi\rangle : \text{out } q} \quad \frac{q \neq q'}{[q']_r : \text{out } q} \\
\ell \xrightarrow{r} v : \text{out } q \quad \frac{\ell \neq \ell'}{\ell' \xrightarrow{r} v : \text{out } \ell}
\end{array}$$

Basic proof rules for quantum ownership.

$$\begin{array}{c}
(\bar{q}, \bar{q}') \mapsto |\psi\rangle |\psi'\rangle \dashv\vdash \bar{q} \mapsto |\psi\rangle * \bar{q}' \mapsto |\psi'\rangle \quad \text{QPTTO-TENSOR} \\
\bar{q} \xrightarrow{r+r'} |\psi\rangle \dashv\vdash \bar{q} \xrightarrow{r} |\psi\rangle * \bar{q} \xrightarrow{r'} |\psi\rangle \quad \bar{q} \xrightarrow{r} |\psi\rangle \vdash (\bar{q} \text{ are pairwise distinct}) \wedge r \leq 1 \\
\frac{\{\bar{q}\} \cap \{\bar{q}'\} \neq \emptyset}{\bar{q} \xrightarrow{r} |\psi\rangle * \bar{q}' \xrightarrow{r'} |\psi'\rangle \vdash \bar{q} = \bar{q}' \wedge |\psi\rangle = |\psi'\rangle} \quad \text{QPTTO-AGREE} \\
(q_{\sigma(1)}, q_{\sigma(2)}, \dots, q_{\sigma(n)}) \xrightarrow{r} \ulcorner \sigma^{-1} \urcorner |\psi\rangle \dashv\vdash (q_1, q_2, \dots, q_n) \xrightarrow{r} |\psi\rangle \quad \text{QPTTO-PERMUTE} \\
[q]_{r+r'} \dashv\vdash [q]_r * [q]_{r'} \quad [q]_r \vdash r \leq 1 \\
\{\text{emp}\} \text{qalloc } \{q. q \mapsto |0\rangle * [q]_1\} \quad \{q \mapsto |0\rangle * [q]_1\} \text{qfree } q \{\text{emp}\} \\
\{(\bar{q}, \bar{q}') \mapsto |\psi\rangle\} U[\bar{q}] \{(\bar{q}, \bar{q}') \mapsto (U_{\bar{q}} \otimes I_{\bar{q}'}) |\psi\rangle\}
\end{array}$$

The rule **QPTTO-TENSOR** cannot be applied to fractional points-to tokens, because that would violate the agreement rule **QPTTO-AGREE**, due to tensor decomposition $|\varphi\rangle = |\psi\rangle |\psi'\rangle$ of a vector $|\varphi\rangle$ not being unique. In the rule **QPTTO-PERMUTE**, σ is a permutation (bijective map) over $\{1, 2, \dots, n\}$ and ${}^r\sigma^r$ denotes the permutation matrix over $(\mathbb{C}^2)^{\otimes n}$ mapping $|b_1 b_2 \dots b_n\rangle$ to $|b_{\sigma(1)} b_{\sigma(2)} \dots b_{\sigma(n)}\rangle$ (where $b_i \in \{0, 1\}$).

Proof rules for heap ownership.

$$\begin{array}{c} \ell \xrightarrow{r+r'} v \dashv\vdash \ell \xrightarrow{r} v * \ell \xrightarrow{r'} v \quad \ell \xrightarrow{r} v \vdash r \leq 1 \\ \{\text{emp}\} \text{mkref } v \{ \ell. \ell \xrightarrow{r} v \} \quad \{ \ell \xrightarrow{r} v \} !\ell \{ v'. v = v' \wedge \ell \xrightarrow{r} v \} \quad \{ \ell \xrightarrow{r} v \} \ell \leftarrow v' \{ \ell \xrightarrow{r} v' \} \end{array}$$

Proof rules for quantum superposition.

$$\begin{array}{c} \frac{\{ \bar{q} \xrightarrow{r} |\psi\rangle * P \} e \{ \bar{q} \xrightarrow{r} |\varphi\rangle * Q \} \quad \{ \bar{q} \xrightarrow{r} |\psi'\rangle * P \} e \{ \bar{q} \xrightarrow{r} |\varphi'\rangle * Q \} \quad P, Q: \text{precise}}{\{ \bar{q} \xrightarrow{r} (\alpha |\psi\rangle + \beta |\psi'\rangle) * P \} e \{ \bar{q} \xrightarrow{r} (\alpha |\varphi\rangle + \beta |\varphi'\rangle) * Q \} \quad P, Q: \text{precise}} \quad \text{QPTTO-LINCOMB} \\ \frac{P: \text{out } \bar{q} \quad P, Q: \text{precise} \quad \{ \bar{q} \xrightarrow{r} |\psi\rangle * P \} e \{ \bar{q} \xrightarrow{r} |\psi\rangle * Q \} \quad \{ \bar{q} \xrightarrow{r} |\psi'\rangle * P \} e \{ \bar{q} \xrightarrow{r} |\psi'\rangle * Q \}}{\{ \bar{q} \xrightarrow{r} (\alpha |\psi\rangle + \beta |\psi'\rangle) * P \} e \{ \bar{q} \xrightarrow{r} (\alpha |\psi\rangle + \beta |\psi'\rangle) * Q \}} \quad \text{FRQPTTO-LINCOMB} \end{array}$$

Proof rules for modifying fractional quantum points-to tokens.

$$\begin{array}{c} \frac{\{ \bar{q} \xrightarrow{r} |\psi\rangle * P \} e \{ v. \bar{q} \xrightarrow{r} |\psi\rangle * Q_v \} \quad P: \text{out } \bar{q} \quad r < 1}{\{ \bar{q} \xrightarrow{r'} |\psi\rangle * P \} e \{ v. \bar{q} \xrightarrow{r'} |\psi\rangle * Q_v \}} \\ \frac{\{ \bar{q} \xrightarrow{r} |\psi\rangle * \bar{q}' \xrightarrow{r} |\psi'\rangle * P \} e \{ v. \bar{q} \xrightarrow{r} |\psi\rangle * \bar{q}' \xrightarrow{r} |\psi'\rangle * Q_v \} \quad P: \text{out } \bar{q}, \bar{q}'}{\{ (\bar{q}, \bar{q}') \xrightarrow{r} |\psi\rangle |\psi'\rangle * P \} e \{ v. (\bar{q}, \bar{q}') \xrightarrow{r} |\psi\rangle |\psi'\rangle * Q_v \}} \\ \frac{\{ (\bar{q}, \bar{q}') \xrightarrow{r} |\psi\rangle |\psi'\rangle * P \} e \{ v. (\bar{q}, \bar{q}') \xrightarrow{r} |\psi\rangle |\psi'\rangle * Q_v \} \quad P: \text{out } \bar{q}, \bar{q}'}{\{ \bar{q} \xrightarrow{r} |\psi\rangle * \bar{q}' \xrightarrow{r} |\psi'\rangle * P \} e \{ v. \bar{q} \xrightarrow{r} |\psi\rangle * \bar{q}' \xrightarrow{r} |\psi'\rangle * Q_v \}} \end{array}$$

Proof rules for promotion by atomicity.

$$\begin{array}{c} \frac{e \text{ is atomic} \quad P: \text{out } \bar{q} \quad \{ \bar{q} \xrightarrow{r} |\psi\rangle * [\bar{q}]_1 * P \} e \{ v. \bar{q} \xrightarrow{r} |\psi\rangle * [\bar{q}]_1 * Q_v \}}{\{ \bar{q} \xrightarrow{r} |\psi\rangle * P \} e \{ v. \bar{q} \xrightarrow{r} |\psi\rangle * Q_v \}} \quad \text{QPTTO-PROMOTE} \\ \frac{e \text{ is atomic} \quad P: \text{out } q \quad \forall |\psi\rangle. \{ q \xrightarrow{r} |\psi\rangle * [q]_1 * P \} e \{ v. q \xrightarrow{r} |\psi\rangle * [q]_1 * Q_v \}}{\{ [q]_r * P \} e \{ v. [q]_r * Q_v \}} \quad \text{QTOK-PROMOTE} \\ \frac{e \text{ is atomic} \quad P: \text{out } \ell \quad \{ \ell \xrightarrow{r} v * P \} e \{ v'. \ell \xrightarrow{r} v * Q_{v'} \}}{\{ \ell \xrightarrow{r} v * P \} e \{ v'. \ell \xrightarrow{r} v * Q_{v'} \}} \quad \text{HPTTO-PROMOTE} \end{array}$$

C Semantic Model of Our Separation Logic

Here we present the semantic model of our separation logic presented in §4.

Heap PCM. The heap PCM (partial commutative monoid) **HEAP** is defined in a standard way:

$$\begin{aligned} |\text{HEAP}| &\triangleq \text{Loc} \xrightarrow{\text{fin}} (0, 1] \times \text{Val} & \varepsilon_{\text{HEAP}} &\triangleq \emptyset \\ \hat{H} \cdot_{\text{HEAP}} \hat{H}' &\triangleq \lambda \ell \in \text{dom } \hat{H} \cup \text{dom } \hat{H}'. \begin{cases} \hat{H}[\ell] & \ell \notin \text{dom } \hat{H}' \\ \hat{H}'[\ell] & \ell \notin \text{dom } \hat{H} \\ \hat{H}[\ell] \cdot \hat{H}'[\ell] & \text{otherwise} \end{cases} \end{aligned}$$

$$\text{where } (q, v) \cdot (q', v') \triangleq \begin{cases} (q + q', v) & q + q' \leq 1, v = v' \\ \text{undefined} & \text{otherwise} \end{cases}$$

The lifting $\ulcorner H \urcorner \in |\text{HEAP}|$ of a heap $H \in \text{Heap}$ into a heap PCM element is defined as follows:

$$\ulcorner H \urcorner \triangleq \lambda \ell \in \text{dom } H. (1, H[\ell])$$

Quantum memory PCM. The qubit token PCM QTOK is defined as follows:

$$|\text{QTOK}| \triangleq \text{Qname} \xrightarrow{\text{fin}} [0, 1] \quad \varepsilon_{\text{QTOK}} \triangleq \lambda _ . 0 \quad R \cdot_{\text{QTOK}} R' \triangleq \lambda q. \begin{cases} Rq + R'q & Rq + R'q \leq 1 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Here, by $\xrightarrow{\text{fin}}$, we mean a finite-support mapping. For any $R \in |\text{QTOK}|$, the support $\text{supp } R \triangleq \{q \in \text{Qname} \mid Rq \neq 0\}$ is a finite set.

The quantum points-to token PCM QPTTO is defined as follows:

$$|\text{QPTTO}| \triangleq \coprod_{qs \in \text{Pow}_{\text{fin}} \text{Qname}} (\mathbb{C}^2)^{\otimes qs} \times \coprod_{qss \in \text{QnamePart } qs} (0, 1)^{qss} \times \coprod_{qs' \in qss} (\mathbb{C}^2)^{\otimes qs'}$$

where $\text{QnamePart } qs \triangleq \{ \{\overline{qs'}\} \in \text{Pow}_{\text{fin}} \text{Pow}_{\text{fin}} \text{Qname} \mid qs, \overline{qs'} \text{ are pairwise disjoint} \}$

$$\varepsilon_{\text{QPTTO}} \triangleq (\emptyset, 1, \emptyset, \emptyset, \emptyset)$$

$$(qs, |\psi\rangle, qss, R, F) \cdot_{\text{QPTTO}} (qs', |\psi'\rangle, qss', R', F') \triangleq \begin{cases} \text{normal}_{\text{QPTTO}}(qs \cup qs', |\psi\rangle \otimes |\psi'\rangle, & qs, qs', (qss \cup qss') \text{'s elements are pairwise disjoint,} \\ qss \cup qss', (\lambda qs_+. Rqs_+ + R'qs_+, F \cup F') & F|_{qss \cap qss'} = F'|_{qss \cap qss'}, \forall qs_+. Rqs_+ + R'qs_+ \leq 1 \\ \text{undefined} & \text{otherwise} \end{cases}$$

where $\text{normal}_{\text{QPTTO}}(qs, |\psi\rangle, qss, R, F) \triangleq \text{let } qss' = \{qs' \in qss \mid Rqs' = 1\} \text{ in}$

$$(qs \cup \bigcup qss', |\psi\rangle \otimes \bigotimes_{qs' \in qss'} Fqs', qss \setminus qss', R|_{qss \setminus qss'}, F|_{qss \setminus qss'})$$

Here we introduce the function $\text{normal}_{\text{QPTTO}}$ to normalize the state by clearing parts of the full fraction. In the definition of $\cdot_{\text{QPTTO}}$, the application Rqs_+ is defined as 0 if $qs_+ \notin \text{dom } R$. We write $\hat{M}$ for elements of $|\text{QPTTO}|$.

The quantum memory PCM QMEM is defined as the product of the quantum points-to token PCM and the qubit token PCM:

$$\text{QMEM} \triangleq \text{QPTTO} \times \text{QTOK}$$

We write $\hat{M}$ for elements of $|\text{QMEM}|$. The lifting $\ulcorner M \urcorner \in |\text{QMEM}|$ of a quantum memory $M \in \text{Qmem}$ into a quantum memory PCM element is defined as follows:

$$\ulcorner M \urcorner \triangleq ((qs, |\psi\rangle, \emptyset, \emptyset, \emptyset), \lambda q. \text{if } q \in qs \text{ then } 1 \text{ else } 0)$$

Global state PCM. The global state PCM is the product of the heap PCM and quantum memory PCM:

$$\text{GLOB} \triangleq \text{QMEM} \times \text{HEAP}$$

We write $\hat{G}$ for elements of $|\text{GLOB}|$. The lifting $\ulcorner G \urcorner \in |\text{GLOB}|$ of a global state $G \in \text{Glob}$ into a global state PCM element is defined as follows:

$$\ulcorner (M, H) \urcorner \triangleq (\ulcorner M \urcorner, \ulcorner H \urcorner)$$

Propositions. We interpret a proposition $P \in \text{SLProp}$ as a predicate over the global PCM $\llbracket P \rrbracket : |\text{GLOB}| \rightarrow \text{Prop}$ as follows:

$$\begin{aligned} \llbracket P \rrbracket _ &\triangleq p & \llbracket P \wedge Q \rrbracket \hat{G} &\triangleq \llbracket P \rrbracket \hat{G} \wedge \llbracket Q \rrbracket \hat{G} & \llbracket P \vee Q \rrbracket \hat{G} &\triangleq \llbracket P \rrbracket \hat{G} \vee \llbracket Q \rrbracket \hat{G} \\ \llbracket P \rightarrow Q \rrbracket \hat{G} &\triangleq \llbracket P \rrbracket \hat{G} \rightarrow \llbracket Q \rrbracket \hat{G} & \llbracket \forall x. P_x \rrbracket \hat{G} &\triangleq \forall x. \llbracket P_x \rrbracket \hat{G} & \llbracket \exists x. P_x \rrbracket \hat{G} &\triangleq \exists x. \llbracket P_x \rrbracket \hat{G} \\ \llbracket \text{emp} \rrbracket \hat{G} &\triangleq \hat{G} = \varepsilon & \llbracket P * Q \rrbracket \hat{G} &\triangleq \exists \hat{G}_1, \hat{G}_2. \hat{G} = \hat{G}_1 \cdot \hat{G}_2 \wedge \llbracket P \rrbracket \hat{G}_1 \wedge \llbracket Q \rrbracket \hat{G}_2 \end{aligned}$$

$$\begin{aligned}
\llbracket P \multimap Q \rrbracket \hat{G} &\triangleq \forall \hat{G}' \text{ s.t. } \hat{G} \cdot \hat{G}' \downarrow . \llbracket P \rrbracket \hat{G}' \rightarrow \llbracket Q \rrbracket (\hat{G} \cdot \hat{G}') \\
\llbracket \bar{q} \xrightarrow{r} |\psi\rangle \rrbracket ((\dot{M}, R), \hat{H}) &\triangleq R = \varepsilon \wedge \hat{H} = \varepsilon \wedge \\
&(\bar{q} \text{ are pairwise distinct}) \wedge r \leq 1 \wedge \dot{M} = \text{qptto}^r(\{\bar{q}\}, \ulcorner \bar{q}^\top |\psi\rangle) \\
\text{where } \text{qptto}^r(qs, |\psi\rangle) &\triangleq \begin{cases} (qs, |\psi\rangle, \emptyset, \emptyset, \emptyset) & r = 1 \\ (\emptyset, 1, \{qs\}, (\lambda_{-}. r), (\lambda_{-}. |\psi\rangle)) & r < 1 \end{cases} \\
\llbracket [q]_r \rrbracket ((\dot{M}, R), \hat{H}) &\triangleq \dot{M} = \varepsilon \wedge \hat{H} = \varepsilon \wedge Rq = r \wedge \forall q' \neq q. Rq' = 0 \\
\llbracket \ell \xrightarrow{r} v \rrbracket (\hat{M}, \hat{H}) &\triangleq \hat{M} = \varepsilon \wedge \hat{H} = \{(\ell, (r, v))\}
\end{aligned}$$

In the semantics of $\bar{q} \xrightarrow{r} |\psi\rangle$, we write $\ulcorner \bar{q}^\top$ for the linear map that maps $|b_1 b_2 \cdots b_n\rangle$ to $|\{q_i \mapsto b_i \mid i\}\rangle$, under the condition that $\bar{q} = q_1, q_2, \dots, q_n$ are pairwise distinct.

Basic judgments. The entailment judgment is interpreted as follows, as usual:

$$\llbracket P \vdash Q \rrbracket \triangleq \forall \hat{G}. \llbracket P \rrbracket \hat{G} \rightarrow \llbracket Q \rrbracket \hat{G}$$

The exactness judgment is interpreted as follows:

$$\llbracket P : \text{precise} \rrbracket \triangleq \forall \hat{G} \text{ s.t. } \llbracket P \rrbracket \hat{G}. \forall \hat{G}' \text{ s.t. } \llbracket P \rrbracket \hat{G}'. \hat{G} = \hat{G}'$$

The ownership exclusion judgment is interpreted as follows:

$$\llbracket P : \text{out } q \rrbracket \triangleq \forall \hat{M}, \hat{H}. \llbracket P \rrbracket (\hat{M}, \hat{H}) \rightarrow q \notin \text{dom } \hat{M} \quad \llbracket P : \text{out } \ell \rrbracket \triangleq \forall \hat{M}, \hat{H}. \llbracket P \rrbracket (\hat{M}, \hat{H}) \rightarrow \ell \notin \text{dom } \hat{H}$$

Here, we define $\text{dom } \hat{M} \subseteq \text{Qname}$ as follows:

$$\text{dom}((qs, |\psi\rangle, qss, R', F), R) \triangleq qs \cup \bigcup qss \cup \text{supp } R$$

Hoare triple. The Hoare triple $\{P\} e \{v. Q_v\}$ is interpreted as follows:

$$\llbracket \{P\} e \{v. Q_v\} \rrbracket \triangleq \forall \hat{G}. \llbracket P \rrbracket \hat{G} \rightarrow \text{Hoare}(e, \hat{G}, \lambda v. \llbracket Q_v \rrbracket)$$

Here, the predicate $\text{Hoare} : \text{Exp} \times |\text{GLOB}| \times (\text{Val} \rightarrow |\text{GLOB}| \rightarrow \text{Prop}) \rightarrow \text{Prop}$ is coinductively defined as follows:

$$\begin{aligned}
\text{Hoare}(e, \hat{G}, \Phi) &\triangleq_v (e \in \text{Val} \wedge \Phi e \hat{G}) \vee \forall G, \hat{G}_+ \text{ s.t. } \ulcorner G^\top = \hat{G} \cdot \hat{G}_+ \\
&\text{red}(e, G) \wedge \forall (e', G') \leftarrow (e, G). \exists \hat{G}' \text{ s.t. } \ulcorner G'^\top = \hat{G}' \cdot \hat{G}_+. \text{Hoare}(e', \hat{G}', \Phi)
\end{aligned}$$